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Cycles containing many vertices of large degree

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Abstract

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Let G be a 2-connected graph of order n, r a real number and $V_r = \{v \in V(G) \mid d(v) \geq r\}$. It is shown that G contains a cycle missing at most $\max\{0, n-2r\}$ vertices of V_r, yielding a common generalization of a result of Dirac and one of Shi Ronghua. A stronger conclusion holds if $r \geq \frac{1}{3}(n+2)$: G contains a cycle C such that either $V(C) \supseteq V_r$ or $|V(C)| \geq 2r$. This theorem extends a result of Häggkvist and Jackson and is proved by first showing that if $r \geq \frac{1}{3}(n+2)$, then G contains a cycle C for which $|V_r \cap V(C)|$ is maximal such that $N(x) \subseteq V(C)$ whenever $x \in V_r - V(C)$ (*). A result closely related to (*) is stated and a result of Nash-Williams is extended using (*).

1. Preliminaries

We use [1] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph, S a subset of $V(G)$ and C a cycle of G. For a real number r, we denote by $V_r(G)$, or just V_r, the set $\{v \in V(G) \mid d(v) \geq r\}$. The cycle C is called *S-longest* if $|S \cap V(C)| \geq |S \cap V(C')|$ for every cycle C' of G. The cycle C is *S-dominating* if every vertex in $S - V(C)$ has all its neighbors on C. We denote by $\vec{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $S \subseteq V(C)$, then $S^+ = \{w^+ \mid w \in S\}$.

Analogous terminology and notation is used with respect to paths instead of cycles.

2. Results

A classical result of Dirac is the following.

Theorem 1 [2]. *If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{1}{2}n$, then G is hamiltonian.*

Dirac also proved the following extension of Theorem 1.

Theorem 2 [2]. *If G is a 2-connected graph of order n, then G contains a cycle of length at least* $\min\{n, 2\delta(G)\}$.

Recently, Shi Ronghua generalized Theorem 1 as follows.

Theorem 3 [6]. *If G is a 2-connected graph of order n, then there exists a cycle in G containing all vertices of degree at least $\frac{1}{2}n$.*

Here we first present a common generalization of Theorems 2 and 3. Its proof is a variation of the proof of Theorem 2 given in [4] (page 393).

Theorem 4. *Let G be a 2-connected graph of order n and r a real number. Then G contains a cycle missing at most* $\max\{0, n-2r\}$ *vertices of V_r.*

Proof. Let $P = x_0x_1 \cdots x_m$ be a V_r-longest path such that $x_0, x_m \in V_r$. Then

$$(N(x_0) \cup N(x_m)) \cap V_r \subseteq V(P). \tag{1}$$

If G contains a cycle C with $V(C) \supseteq V(P)$, then $V(C) \supseteq V_r$, otherwise we easily contradict the choice of P. Hence we may assume

$$\text{no cycle of } G \text{ contains all vertices of } P. \tag{2}$$

We distinguish two cases.

Case 1: $\max\{k \mid x_0x_k \in E(G)\} > \min\{k \mid x_kx_m \in E(G)\}$.

Choose i and j such that $i<j$, $x_0x_j \in E(G)$, $x_ix_m \in E(G)$ and $j-i$ is minimal. Let $C = x_0x_1 \cdots x_ix_mx_{m-1} \cdots x_jx_0$. By (2), the sets $\{x_m\}$, $N(x_m) \cap V(P)$ and $\{x_{k-1} \mid x_k \in N(x_0) \cap V(P),\ k \neq j\}$ are pairwise disjoint subsets of $V(C)$. Since the last-mentioned set has cardinality $|N(x_0) \cap V(P)| - 1$, we obtain

$$|V(C)| \geq |N(x_0) \cap V(P)| + |N(x_m) \cap V(P)|. \tag{3}$$

Set $R = V(G) - V(P)$, $S = V(P) - V(C)$, $R_r = R \cap V_r$, $S_r = S \cap V_r$. By (1), no vertex of R_r is adjacent to either x_0 or x_m, while by (2), no vertex of $R - R_r$ is adjacent to both x_0 and x_m. Hence

$$|N(x_0) \cap R| + |N(x_m) \cap R| \leq |R| - |R_r|. \tag{4}$$

Summing (3) and (4) we obtain

$$2r \leq d(x_0) + d(x_m) \leq |V(C)| + |R| - |R_r|$$
$$= |V(P)| - |S| + |R| - |R_r| = n - (|S| + |R_r|),$$

whence

$$|V_r - V(C)| = |R_r \cup S_r| = |R_r| + |S_r| \leq |R_r| + |S| \leq n - 2r,$$

settling Case 1.

Case 2: $\max\{k \mid x_0x_k \in E(G)\} \leq \min\{k \mid x_kx_m \in E(G)\}$.

Set $i = \max\{k \mid x_0x_k \in E(G)\}$ and $j = \min\{k \mid x_kx_m \in E(G)\}$. Since G is 2-connected, there are two disjoint paths P_1 and P_2 connecting the cycles $x_0x_1 \cdots x_ix_o$ and $x_jx_{j+1} \cdots x_mx_j$. (If $i = j$, then the trivial path with vertex x_i is considered to be one of these paths). Let U be the set of endvertices of P_1 and P_2. ($|U| = 4$ unless $i = j$). We may assume $x_i, x_j \in U$. Possibly, x_i and x_j are ends of the same path. Furthermore, U may contain x_0 and/or x_m. In all possible cases we easily find a cycle C containing $x_o, x_m, N(x_0) \cap V(P)$ and $N(x_m) \cap V(P)$. By (1), $N(x_0) \cap V_r \subseteq N(x_0) \cap V(P) \subseteq V(C)$ and $N(x_m) \cap V_r \subseteq N(x_m) \cap V(P) \subseteq V(C)$. Hence no vertex of $V_r - V(C)$ is adjacent to either x_0 or x_m. By (2) and the hypothesis of Case 2, $|N(x_0) \cap N(x_m)| \leq 1$. Since $x_0, x_m \notin N(x_0) \cup N(x_m)$, we conclude that

$$2r - 1 \leq d(x_0) + d(x_m) - 1 \leq |N(x_0) \cup N(x_m)| \leq n - 2 - |V_r - V(C)|,$$

whence $|V_r - V(C)| < n - 2r$. □

Theorems 2 and 3 are the special cases $r = \delta(G)$ and $r = \frac{1}{2}n$ of Theorem 4, respectively.

Theorem 4 is not best possible in general. For example, if $|V_r(G)| + 2 \leq r = \frac{1}{3}(|V(G)| + 2)$, then the theorem gives nothing, whereas by Corollary 11 below some cycle of G contains all vertices of V_r.

Suppose G is a 2-connected graph of order n and r a real number such that no cycle of G contains all vertices of V_r. Then Theorem 4 asserts that some cycle of G misses at most $n - 2r$ vertices of V_r. By Theorem 7 below, a stronger conclusion holds if $r \geq \frac{1}{3}(n + 2)$: some cycle of G misses at most $n - 2r$ vertices of $V(G)$. Theorem 7 is an easy consequence of the following result, the proof of which is based on ideas from [7].

Theorem 5. *Let G be a 2-connected graph of order n and r a real number with $r \geq \frac{1}{3}(n + 2)$. Then G contains a V_r-longest cycle which is V_r-dominating.*

Proof. Assume no V_r-longest cycle of G is V_r-dominating. Consider a cycle C and a path P satisfying the following requirements:

$$C \text{ is a } V_r\text{-longest cycle.} \tag{5}$$

Subject to (5), $|M(C)|$ is minimal, where $M(C)$ denotes the set of all edges of $G - V(C)$ which are incident with at least one vertex of V_r. (By assumption, $M(C) \neq \emptyset$). (6)

P connects two vertices v_1 and v_2 of C, is internally disjoint from C and contains a vertex $x_0 \in V_r$ incident with an edge of $M(C)$. (7)

Subject to (5), (6) and (7), $|V(P)|$ is minimal. (8)

Subject to (5), (6), (7) and (8), $d_c(v_1, v_2)$ is minimal. (9)

Set $R = V(G) - V(C)$. Orient C such that $|v_1\vec{C}v_2| \leq |v_2\vec{C}v_1|$, and orient P from v_1 to v_2. By (9), $N(x_0) \cap V(C) \subseteq v_2\vec{C}v_1$. Let $v_3, \ldots, v_t$ be the vertices in $(N(x_0) \cap V(C)) - \{v_1, v_2\}$, occurring on $v_2\vec{C}v_1$ in consecutive order. By (5), the sets $V_r \cap v_1^+\vec{C}v_2^-$ and $V_r \cap v_2^+\vec{C}v_3^-$ (or $V_r \cap v_2^+\vec{C}v_1^-$ if $(N(x_0) \cap V(C)) - \{v_1, v_2\} = \emptyset$) are non-empty. Let u_1 be the first vertex on $v_1^+\vec{C}v_2^-$ such that either $u_1 \in V_r$ or u_1 is adjacent to a vertex $w_1 \in R \cap V_r$. Set $x_1 = u_1$ if $u_1 \in V_r$ and $x_1 = w_1$ otherwise. Define $u_2 \in v_2^+\vec{C}v_1^-$ and x_2 similarly. Note that by the choice of u_1 and u_2, any cycle C' with $V(C') \supseteq V(C) - (v_1^+\vec{C}u_1^- \cup v_2^+\vec{C}u_2^-)$ is a V_r-longest cycle satisfying $M(C') \subseteq M(C)$ and hence $M(C') = M(C)$. We have

$$x_i \neq x_0,\ x_ix_0 \notin E(G) \quad \text{and} \quad N(x_i) \cap N(x_0) \cap R = \emptyset \quad (i = 1, 2), \tag{10}$$

otherwise we contradict (5) or (8). Furthermore, x_1 and x_2 do not coincide. Assuming the contrary, the cycle $v_1\vec{P}v_2\overleftarrow{C}u_1x_1u_2\vec{C}v_1$ contradicts (5). A similar argument shows that

$$x_1v,\ x_2w \notin E(G) \text{ whenever } v \in v_2^+\vec{C}u_2 \cup \{x_2\},\ w \in v_1^+\vec{C}u_1 \cup \{x_1\}. \tag{11}$$

For $i = 3, \ldots, t$, set $u_{i1} = v_i^+$. Set $u_{i2} = u_{i1}^+$ if $N(u_{i1}) \cap R = \emptyset$, otherwise let u_{i2} be an arbitrary vertex in $N(u_{i1}) \cap R$ $(i = 3, \ldots, t)$. We have

$$x_1 \neq u_{i2},\ x_2 \neq u_{i2},\ u_{i2} \neq u_{j2}(i, j \in \{3, \ldots, t\},\ i \neq j),$$

otherwise we contradict (5) as above. Furthermore,

$$x_ku_{im} \notin E(G) \quad (i = 3, \ldots, t;\ k = 1, 2;\ m = 1,2). \tag{12}$$

Assuming the contrary, for fixed k and i we contradict (5) unless $m = 2$, $x_k = u_k$, $u_{i1} \in V_r$ and $u_{i2} = u_{i1}^+$. In that case, however, the V_r-longest cycle $v_1\vec{P}x_0v_i\overleftarrow{C}u_1u_{i2}\vec{C}v_1$ (if $k = 1$) or $v_2\vec{P}x_0v_i\overleftarrow{C}u_2u_{i2}\vec{C}v_2$ (if $k = 2$) contradicts (6).

We make another observation.

$$\text{If } v \in u_1^+\vec{C}v_2^- \text{ and } x_2v \in E(G), \text{ then } x_1v^+ \notin E(G). \tag{13}$$

Assuming the contrary, the cycle $v_1\vec{P}v_2\overleftarrow{C}v^+(x_1)u_1\vec{C}v(x_2)u_2\vec{C}v_1$ (where (x_i) should be ignored if $x_i = u_i$ $(i = 1, 2)$) contradicts (5). Similarly we have the following.

$$\text{If } v \in u_2^+\vec{C}v_1^- \text{ and } x_1v \in E(G), \text{ then } x_2v^+ \notin E(G). \tag{14}$$

Set $U = V(C) \cup \{x_1, x\} \cup \{u_{i2} \mid 3 \leq i \leq t\}$. Define a bijection $\varphi : U \to U$ as follows:

$$\text{If } x_i \neq u_i, \text{ then } \varphi(u_i) = x_i \text{ and } \varphi(x_i) = u_i^+ \ (i = 1, 2). \tag{15}$$

$$\text{If } u_{i2} \notin V(C), \text{ then } \varphi(u_{i1}) = u_{i2} \text{ and } \varphi(u_{i2}) = u_{i1}^+ \ (i = 3, \ldots, t). \tag{16}$$

$$\text{If } \varphi(v) \text{ is not yet defined by (15) or (16), then } \varphi(v) = v^+. \tag{17}$$

Define

$$A_1 = \{v \in u_1 \vec{C} u_2^- \cup \{x_1\} \mid x_1\varphi(v) \in E(G)\},$$

$$A_2 = \{v \in u_1 \vec{C} u_2^- \cup \{x_1\} \mid x_2 v \in E(G)\},$$

$$B_1 = \{v \in u_2 \vec{C} u_i^- \cup \{x_2\} \cup \{u_{i2} \mid 3 \leq i \leq t\} \mid x_1 v \in E(G)\},$$

$$B_2 = \{v \in u_2 \vec{C} u_i^- \cup \{x_2\} \cup \{u_{i2} \mid 3 \leq i \leq t\} \mid x_2\varphi(v) \in E(G)\},$$

$$D_i = \{v \in V(G) - U \mid x_i v \in E(G)\} \ (i = 0, 1, 2).$$

Since $\varphi : U \to U$ is a bijection, we have

$$d(x_i) = |A_i| + |B_i| + |D_i| \quad (i = 1, 2).$$

Furthermore,

$$d(x_o) \leq |D_0| + t.$$

By (10), (11), (13) and (14), the sets $A_1, A_2, B_1, B_2, D_1, D_2, D_3$ are pairwise disjoint. By (10) and (12), the vertices $x_0, u_{31}, \ldots, u_{t1}$ are in none of these sets. Since $x_0, x_1, x_2 \in V_r$, we conclude that

$$\begin{aligned} n + 2 \leq 3r \leq \sum_{i=0}^{2} d(x_i) &\leq \sum_{i=1}^{2} |A_i| + \sum_{i=1}^{2} |B_i| + \sum_{i=0}^{2} |D_i| + t \\ &\leq n - 1 - (t - 2) + t = n + 1. \end{aligned}$$

This contradiction completes the proof. □

The lower bound $\frac{1}{3}(n + 2)$ imposed on r in Theorem 5 cannot be relaxed: in the graph $K_2 \vee 3K_{r-1}$ (and also in suitable spanning subgraphs of this graph), no V_r-longest cycle is V_r-dominating ($r \geq 3$).

As a consequence of Theorem 5, a 2-connected graph G of order n with $\delta(G) \geq \frac{1}{3}(n + 2)$ contains a longest (i.e., $V(G)$-longest) cycle which is a dominating (i.e., $V(G)$-dominating) cycle. Nash-Williams [5] showed that in such a graph in fact every longest cycle is a dominating cycle. The conclusion of Theorem 5 cannot be strengthened correspondingly. To see this, add three new vertices x_1, x_2, x_3 to the complete bipartite graph $K_{r,r+1}$ $(r \geq 6)$ with bipartition $\{\{u_1, \ldots, u_{r+1}\}, \{v_1, \ldots, v_r\}\}$ and join x_i to u_i and v_1 $(i = 1, 2, 3)$. Since $r \geq 6$, the resulting graph G satisfies $r \geq \frac{1}{3}(|V(G)| + 2)$. Yet there exist V_r-longest cycles, even V_r-longest cycles of maximal length, which are not V_r-dominating.

The following result is closely related to Theorem 5.

Theorem 6. *In every 2-connected graph G of order n there exists a cycle containing at least one end of each edge uv with $|N(u) \cup N(v)| \geq \frac{1}{3}(n+5)$.*

Theorem 6 can be proved by considering a cycle C for which $|\{uv \in E(G - V(C))| \ |N(u) \cup N(v)| \geq \frac{1}{3}(n+5)\}|$ is minimal and combining ideas from the proofs of Theorem 5 and [7, Theorem 3]. Theorem 6 implies that under the hypothesis of Theorem 5, G contains a V_r-dominating cycle, but not that G contains a V_r-longest cycle which is V_r-dominating.

We now use Theorem 5 to derive the announced improvement of Theorem 4 for $r \geq \frac{1}{3}(n+2)$.

Theorem 7. *Let G be a 2-connected graph of order n and r a real number with $r \geq \frac{1}{3}(n+2)$. Then G contains a cycle C such that either $V(C) \supseteq V_r$ or $|V(C)| \geq 2r$.*

Proof. By Theorem 5, G contains a V_r-longest cycle $\vec{C}$ which is V_r-dominating. Assuming $V(C) \not\supseteq V_r$, let x be a vertex in $V_r - V(C)$. Then $N(x) \subseteq V(C)$. Clearly, $N(x) \cap N(x)^+ = \emptyset$. Hence $|V(C)| \geq 2|N(x)| \geq 2r$. □

In a sense, Theorem 7 is best possible. Let G be a spanning subgraph of $K_{r,s}$ $(r < s \leq 2r-2)$ such that $|V_r(G)| = 2r+1$, i.e., $G[V_r] \simeq K_{r,r+1}$. Then no cycle of G contains V_r, while every cycle of G has length at most $2r$. Theorem 7 is also best possible in the sense that the lower bound $\frac{1}{3}(n+2)$ imposed on r cannot be relaxed. One easily finds a 2-connected spanning subgraph H of $K_2 \vee 3(K_2 \vee (r-3)K_1) \subseteq K_2 \vee 3K_{r-1}$ $(r \geq 5)$ such that no cycle of H contains $V_r(H)$, while $|V(C)| \leq 8 < 2r$ for any cycle C of H.

An immediate consequence of Theorem 7 is the following result of Häggkvist and Jackson.

Corollary 8 [3]. *Let G be a 2-connected graph of order n and r a real number with $r \geq \frac{1}{3}(n+2)$. If $|V_r| \geq 2r$, then G contains a cycle of length at least $2r$.*

Our next result shows that under a suitable additional condition one is able to distinguish between the two alternatives in the conclusion of Theorem 7.

Theorem 9. *Let G be a 2-connected graph of order n and r a real number with $r \geq \frac{1}{3}(n+2)$. If $\alpha(G[V_r]) \leq r$, then there exists a cycle of G containing all vertices of V_r.*

Proof. By Theorem 5, G contains a V_r-longest cycle $\vec{C}$ which is V_r-dominating. Assuming $V(C) \not\supseteq V_r$, let x be a vertex in $V_r - V(C)$. Then $N(x) \subseteq V(C)$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of x, occurring on $\vec{C}$ in consecutive order. Since C is a V_r-longest cycle, $v_i^+ \vec{C} v_{i+1}^- \cap V_r \neq \emptyset$ $(i = 1, \ldots, k$; indices mod $k)$. Let x_i be the first vertex on $v_i^+ \vec{C} v_{i+1}^-$ that belongs to V_r. Clearly, $\{x, x_1, \ldots, x_k\}$ is an independent set, so $\alpha(G[V_r]) \geq k+1 \geq r+1$. □

The upper bound r imposed on $\alpha(G[V_r])$ in Theorem 9 is tight, as shown by the graph G defined after the proof of Theorem 7. Also, the condition $r \geq \frac{1}{3}(n+2)$ again cannot be relaxed, as shown by the graph $K_2 \vee 3K_{r-1} (r \geq 3)$.

We note that Theorem 3 is not only implied by Theorems 4 and 7, but also by Theorem 9, since $\alpha(G[V_{\frac{1}{2}n}]) \leq \frac{1}{2}n$ for any graph G.

The case $r = \delta(G)$ of Theorem 9 was obtained by Nash-Williams.

Corollary 10 [5]. *If G is a 2-connected graph of order n with $\delta(G) \geq \max\{\frac{1}{3}(n+2), \alpha(G)\}$, then G is hamiltonian.*

Another obvious consequence of Theorem 9 is the following.

Corrollary 11. *Let G be a 2-connected graph of order n and r a real number with $r \geq \frac{1}{3}(n+2)$. If $|V_r| \leq r$, then there exists a cycle of G containing all vertices of V_r.*

We do not believe that the upper bound r on $|V_r|$ in Corollary 11 is tight. It would be interesting to find the best possible upper bound.

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